Answers to Exam I. (Test 1b)

Part I.

1. The x-coordinate of the point on the graph of $y=3x^2-4x+7$ with a horizontal tangent line is

Sol: $y'=6x-4=0 \Rightarrow x=\frac{2}{3}$

2. Let $f(x)=\sqrt{6+x}$ and $g(x)=\frac{x}{x+3}$. Then $f(g(1))$ is

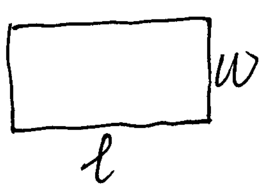
$g(1)=\frac{1}{1+3}=\frac{1}{4}$ $\quad f(g(1))=f(\frac{1}{4})=\sqrt{6+\frac{1}{4}}=\sqrt{\frac{25}{4}}=\frac{5}{2}$

3. The limit $\lim\limits_{x\to -1}\frac{x^2-x-2}{2(x+1)}=\lim\limits_{x\to -1}\frac{(x-2)(x+1)}{2(x+1)}=\lim\limits_{x\to -1}\frac{x-2}{2}=\frac{-3}{2}$

4. Another way of writing $\frac{x^2-4}{|x-2|}$ is

$$\frac{x^2-4}{|x-2|}=\frac{(x+2)(x-2)}{|x-2|}=\begin{cases} x+2 & x>2 \\ -(x+2) & x<2 \end{cases}$$

5. A rectangular sheet plastic has length l and width w. If the length of the perimeter of the sheet is fixed at 6. express the ~~aera~~ aera A as a function of l along.



$2(l+w)=6 \Rightarrow l+w=3 \Rightarrow w=3-l$

$A=A(l)=lw=l(3-l)=-l^2+3l$

PART II

6. (a) Find the equation of the tagent line to the graph of the function $g(x)=-2x^2+8$ at the point $(1,6)$.

Sol: $g'(x)=-4x$ $\quad g'(1)=-4$

the eqn of the tangent line is:

$y-6=-4(x-1)$

i.e. $y=-4x+10$

(b) Find the equation of the line through the point $(1,6)$ with slope $\frac{1}{4}$.

Sol: By slope-point form, we have

$y-6=\frac{1}{4}(x-1)$

i.e. $y=\frac{1}{4}x+5\frac{3}{4}$

(c) Are the lines in ~~th~~ parts (a) and (b), parallel, perpendicular, or neither. Justify your answer.

Ans: They are perpendicular to each other. Since these two lines pass the same point $(1,6)$ with the product of their slops equal to -1. namely $(-4)(\frac{1}{4})=-1$.

7. Find the following limits if they exist for the function.

$$f(x)=\begin{cases} \dfrac{2x}{x^2+1}, & \text{if } x<1 \\ 3x-2, & \text{if } x\geq 1 \end{cases}$$

(a) $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{2x}{x^2+1} = \frac{2}{2} = 1$

(b) $\displaystyle\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3x-2) = 1$

(c) $\displaystyle\lim_{x\to 1} f(x) = 1$ since $\displaystyle\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

(d) Explain whether or not f is continuous at $x=1$.

Note $f(1) = 3\times 1 - 2 = 1 = \displaystyle\lim_{x\to 1} f(x)$, so f is continuous at $x=1$.

8. Compute the following limits.

(a) $$\lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2} = \lim_{x\to 2} \frac{\sqrt{x-1}-1}{x-2}\cdot\frac{\sqrt{x-1}+1}{\sqrt{x-1}+1} = \lim_{x\to 2} \frac{x-1-1}{(x-2)(\sqrt{x-1}+1)}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x-1}+1} = \frac{1}{2}$$

(b) $$\lim_{x\to 0} \frac{\sin 3x}{4x}(5-\cos(2x))$$

$$= \lim_{x\to 0} \frac{\sin 3x}{4x} \lim_{x\to 0}(5-\cos(2x)) = \lim_{x\to 0} \frac{3}{4}\cdot\frac{\sin 3x}{3x} \lim_{x\to 0}(5-\cos(2x))$$

$$= \frac{3}{4}\times 4 = 3$$

9. Apply the definition of the derivative as a limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ to obtain $f'(x)$ for

$f(x) = \dfrac{1}{2x-1}$.

<u>Sol</u>: $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h}$$

$$= \lim_{h\to 0} \frac{2x-1-2(x+h)+1}{h[2(x+h)-1](2x-1)} = \lim_{h\to 0} \frac{-2h}{h[2(x+h-1](2x-1)}$$

$$= \lim_{h\to 0} \frac{-2}{[2(x+h)-1](2x-1)} = \frac{-2}{(2x-1)^2}$$

10. Let $f(x)=\sqrt{3x^2+1}$ and $g(x)=2-5x^3$. Use the intermediate value property of continuous functions to show that the equation $f(x)-g(x)=0$ has at least one solution for x in the interval $[0,1]$.

Sol: Let $F(x)=f(x)-g(x)$

note f, g are both continuous on $[0,1]$, hence F is continuous on $[0,1]$. Now $F(0)=f(0)-g(0)=1-(2)=-1$,

$F(1)=2-(2-5)=5$, we have $F(0)F(1)<0$.

Thus, by IMT, $\exists\, c \in (0,1)$ such that $F(c)=0$, namely $f(c)-g(c)=0$. We have shown c is one ~~of~~ ~~the~~ solution with $0<c<1$.

11. Let $f(x)=\dfrac{x^2-9}{x^2-8x+15}$ What is the domain of ~~the~~ $f(x)$?

We need $x^2-8x+15\neq 0$. i.e $(x-3)(x-5)\neq 0$.

Thus domain of f is $\{x\in\mathbb{R} \mid x\neq 3, ~~and~~\ x\neq 5\}$

or $=\{x\in\mathbb{R} \mid x<3 \text{ or } 3<x<5 \text{ or } x>5\}$

or $=(-\infty,3)\cup(3,5)\cup(5,\infty)$

(b) At which of the points not in the domain of f, call it a, does the limit exist? Compute the limit.

Sol: We note $f(x)=\dfrac{x^2-9}{(x^2-8x+15)}=\dfrac{(x+3)(x-3)}{(x-3)(x-15)}$

Thus $\lim\limits_{x\to 3} f(x)$ exists and

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{(x+3)(x-3)}{(x-3)(x-15)}=\lim_{x\to 3}\frac{x+3}{x-15}=\frac{6}{-12}=-\frac{1}{2}$$

(c) It is possible to define $f(a)$ so that the ftn will be continuous at $x=a$. How should it be defined? Explain why it is continuous now.

Sol: ~~Define~~ ~~$f(a)$~~ $a=3$.

define $f(a)=f(3)=-\frac{1}{2}$.

Then f would be continuous at $a=3$. Since now we have $f(3)=\lim\limits_{x\to 3} f(x)=-\frac{1}{2}$.